Why Are Renormalons Always In The Same Place ?

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Abstract

In order that current and future renormalon results in QCD can be used to their full advantage it is important to understand how the Borel transforms of related functions are themselves related. For example, a change of renormalisation scheme can involve the inversion of a function; how is the transform of a function related to the transform of an inverse ? We outline a method of extracting the dominant behaviour of the Borel transform after the action of simple operations on the function.

1 Introduction

Borel transforms provide a convenient and insightful method of describing the large-order behaviour of QCD perturbation series. An observable

$$R(a) = \sum_{n=0}^{\infty} r_n a^{n+1}, \qquad a \equiv \alpha_s/\pi \tag{1}$$

is represented as the integral

$$R(a) = \int_0^{\infty} F(z) e^{-z/a} dz \tag{2}$$

where $F(z)$ is the transform. (If necessary, principal values are used to define the integral – see Appendix). A singularity of the form

$$F(z) \sim \frac{A}{(1-\alpha z)^{\beta}}, \qquad z \sim 1/\alpha \tag{3}$$

gives rise to a factorial divergence

$$r_n \sim n! A \alpha^n \frac{\beta(\beta+1)\ldots(\beta+n-1)}{n!} \tag{4}$$

in the perturbation expansion of $R(a)$.

Much attention has been devoted to characterising the $F(z)$ of QCD observables, both in general and for particular cases. Arguments related to power corrections show that a typical observable has an infinite chain of singularities of the form (3), equally spaced along the positive real axis of the Borel z-plane [1][2]. These are called IR renormalons and are crucial is assessing the practical reliability of perturbation theory. Although the positions and general form of the renormalons are fixed for any $R(a)$ by these arguments, the particular details (especially the A and β of a singularity) have to be established for each observable separately. Such work has both reproduced the predicted general features and provided specific information about $F(z)$ in cases of interest, see e.g. [3]-[9]. A notable feature of these results is that $F(z)$ is invariably a simpler function than $R(a)$.

However to fully exploit the potential of these results it is often necessary to consider functions related to $R(a)$ rather than the calculated $R(a)$ itself. For example, $R(a)$ and its Borel transform may have been calculated using

the $\overline{MS}$ renormalisation scheme, but it is now the corresponding results in the effective charge scheme [10] that are wanted. The function of interest is now the effective charge β-function

$$\rho(R) \equiv \frac{\beta(a(R))}{da/dR} \tag{5}$$

where $\beta(a)$ is the $\overline{MS}$ β-function. Calculating this involves doing a series of operations to the function $R(a)$: inverting it to give $a(R)$, finding the function-of-a-function $\beta(a(R))$, differentiation and division. In terms of the function these are straightforward enough operations, though possibly difficult to do exactly in practice. A natural question – particularly if, as noted, the observable is simpler as a transform than as a function of a – is what effect these operations have in the Borel plane ? If we know the Borel transform of $R(a)$, what does this tell us about the transform of $a(R)$?

Because such operations are commonplace, questions like this arise continually when applying the results of renormalon calculations. It is this practical need that the method outlined here seeks to meet.

2 The Operations

The operations that this paper will discuss are

1. Multiplication $R_1(a)R_2(a)$
2. Differentiation dR/da
3. Function-of-Function $R_1(R_2(a))$
4. Division $R_1(a)/R_2(a)$
5. Inversion $a(R)$

Some of these are easy and some are hard. The first two are the straightforward ones. Like with any simple integral transform, the Borel transform of a product is given by a convolution integral over the transforms of the factors. If $R_1(a)$ and $R_2(a)$ have transforms $F_1(z)$ and $F_2(z)$, then $R_1(a)R_2(a)$ has a transform

$$F_3(z) = \int_0^z F_1(u)F_2(z-u)du. \tag{6}$$

This will be true even if the transforms contain renormalon singularities (see Appendix). As noted by 't Hooft [11], the presence of such singularities in the factors leads to similar singularities in the product as a result of (6). This aspect is discussed in more detail below. Indeed much of this paper can be considered a generalization of this observation about the convolution integral.

The other straightforward operation is differentiation, since the transform of $a^n d^n R/da^n$ is

$$\frac{d}{dz}\left(z^n \frac{d^{n-1}F}{dz^{n-1}}\right). \tag{7}$$

It is possible to remove powers of a from $a^n d^n R/da^n$ by using

$$\frac{R(a)}{a} = F(0) + \int_0^\infty e^{-z/a} F'(z) dz. \tag{8}$$

The $F(0)$ piece here corresponds to a delta function term in the transform which we will seek to avoid. Thus (8) will only be used in this paper on functions for which $F(0) = 0$.

The remaining operations 3-5 are all intrinsically difficult ones. By this we mean that it is unlikely that closed-form results will ever be found for their effect on transforms, in the way that (6) and (7) exist for multiplication and differentiation. Even with a simple example like $F(z) = 1/(1-z)$, it appears that they can produce a transform that is much more complicated in its fine detail than the input one. Our simple and natural question does not have a simple and natural full answer. However the issue of practicality remains and for this a complete answer to the problem is not necessary. For the effective use of current and foreseeable QCD calculations it will be sufficient to extract only the more important effects that the use of these operations will have on the transforms. Furthermore, by concentrating on the main effects, a simple and natural picture does emerge.

The general question of the effect of the operations 3-5 on Borel summable functions has been addressed before, in a pair of papers by Auberson and Mennessier [12]. Their main conclusion is that the functions thus produced are also Borel summable. This result is clearly a necessary starting point for our investigation in that it establishes that the transforms we will be trying to describe actually exist. In general [12] provides a firm foundation of rigour that will be relied upon implicitly throughout what follows. As

a consequence of the emphasis on extracting information that can be used in practical QCD contexts, the approach and language used here is however very different than in [12].

3 The λ-Expansion

As a concrete example, consider operation 5, inverting a function $R(a)$ to $a(R)$.

The central idea of our approach is to introduce the following split in the initial function

$$R(a) = a + \lambda \tilde{R}(a). \tag{9}$$

The parameter λ is a bookkeeping device which will be set equal to one at the end. Thus

$$\tilde{R}(a) = \sum_{n=1}^{\infty} r_n a^{n+1} \tag{10}$$

is simply all of $R(a)$ except for its lowest order term. Of course (9) really defines a function $R(a, \lambda)$ such that $R(a) = R(a, 1)$. And if we are starting from a QCD observable, $R(a)$ will be renormalisation scheme invariant, but $R(a, \lambda \neq 1)$ will not be. But these nuances are unproblematical and the λ-dependence of $R(a, \lambda)$ will be suppressed in the notation.

It is also possible to think of $a(R)$ as now having a λ-dependence, though one more complicated than that in (9). Expand $a(R)$ as an expansion in λ.

$$a(x) = x + \lambda \frac{\partial a}{\partial \lambda}\bigg|_{\lambda=0} + \frac{\lambda^2}{2} \frac{\partial^2 a}{\partial \lambda^2}\bigg|_{\lambda=0} + \ldots \tag{11}$$

The coefficients are functions of x and can be found by repeatedly differentiating (9) rewritten as

$$x = a(x) + \lambda \tilde{R}(a(x)) \tag{12}$$

with respect to λ. Thus the first differentiation gives

$$0 = \frac{\partial a}{\partial \lambda} + \tilde{R}(a) + \lambda \frac{\partial \tilde{R}}{\partial a} \frac{\partial a}{\partial \lambda} \tag{13}$$

and so

$$\frac{\partial a}{\partial \lambda}\bigg|_{\lambda=0} = -\tilde{R}(x). \tag{14}$$

The first few terms of (11) are found to be

$$
\begin{aligned}
a(x) = x \quad &- \quad \lambda\tilde{R}(x) + \lambda^2\tilde{R}(x)\frac{d\tilde{R}}{dx} \\
&- \quad \frac{\lambda^3}{2}\left(\tilde{R}^2\frac{d^2\tilde{R}}{dx^2} + 2\tilde{R}\left(\frac{d\tilde{R}}{dx}\right)^2\right) + \ldots
\end{aligned} \tag{15}
$$

This will be called the λ-expansion of $a(R)$.

Given that all the other perturbative expansions in this paper are divergent, it is important to emphasize that the λ-expansion is usually convergent as a series in λ. A detailed discussion of this issue is deferred [13], but heuristic arguments can be offered for (15) being convergent. First, note that so far nothing has been assumed about the convergence properties of (10). $\tilde{R}(a)$ can be as favourable and well-behaved a function as required and (15) will still be derived. As a relation between $\tilde{R}(x)$ and $a(x)$ as functions, the structure of the λ-expansion is independent of their behaviour as expansions in x. In this light, there is no *a priori* reason to expect (15) to be especially problematic for the cases of interest. Secondly, when $\tilde{R}(x)$ is divergent, the resulting divergence of $a(x)$ as a series in x can be thought of as having been absorbed into the $\tilde{R}(x)$ that appear in (15) to leave a convergent expression. Thirdly, consider the function

$$
\lambda(y) = \lambda(y, x) \equiv \frac{y}{\tilde{R}(x-y)} \tag{16}
$$

and its inverse $y(\lambda) = y(\lambda, x)$ for a fixed x. $\tilde{R}(a)$ is expected to have one cut in the complex a-plane, along the negative real axis. Thus $\tilde{R}(x-y)$ is analytic about $y = 0$ for all $x > 0$ and $\lambda(y)$ will converge for $|y| < x$. Hence its inverse, $\lambda(y)$, has a non-zero radius of convergence [14], but since

$$
a(x) = x - y(\lambda, x), \tag{17}
$$

so does (15).

Clearly none of these arguments is sufficient to prove that the λ-expansion is convergent for $\lambda = 1$, as required. Pending the detailed discussion [13], the issue is set aside. For present purposes, we merely note that in the results presented here the expansions actually have infinite radii of convergence.

Turning to the use of (15), for a given $R(a)$ and hence $\tilde{R}(a)$, setting $\lambda = 1$ in (15) provides a systematic means of calculating $a(R)$ from $\tilde{R}(a)$. It is a

method of inverting functions. Its importance as such is that it serves to reduce the hard operation of inversion to a sequence (albeit infinite) of easy ones, namely differentiation and multiplication. Given the Borel transform of $R(a)$, (6) - (8) can be used to calculate the Borel transform of any particular term in the λ-expansion and then these contributions summed to obtain the Borel transform of $a(R)$.

4 Calculating the Transform

The simplest case is where the transform of $R(a)$ (and thus $\tilde{R}(a)$) has a single singularity at $z = 1/\alpha$ such that

$$F(z) \sim \frac{A}{(1-\alpha z)^{\beta}}, \qquad z \sim 1/\alpha. \tag{18}$$

The generalization to multiple poles, as required for realistic QCD examples, will be straightforward. Neglecting numerical factors, the general term in (15) is

$$\lambda^n \frac{d^{q_1}\tilde{R}}{dx^{q_1}} \cdots \frac{d^{q_n}\tilde{R}}{dx^{q_n}} \tag{19}$$

$$q_1 + \ldots q_n = n - 1. \tag{20}$$

What is the transform of this, given an $\tilde{R}$ implied by (18) ? Using (7)-(8) this is easily found in principle, but for any particular $F(z)$ that accords with (18) the convolution integrals quickly become impossible to evaluate exactly. It becomes necessary to follow only the more important features of the transforms through the calculation.

Note that if $F(z)$ in (18) has another singularity at $z = 1/\alpha$ with $\beta' = \beta - 1$, then according to (4) the additional contribution to the factorial divergence of the coefficient r_n is suppressed by $1/n$ and so can be neglected at large orders. All singularities at $z = 1/\alpha$ with smaller β are similarily sub-leading in r_n. The singularity with largest β at $z = 1/\alpha$ will be called the dominant singularity, the others sub-dominant. It is the dominant singularities (possibly at different $z = 1/\alpha$) that will be most important in practice. Furthermore, for the operations 1-5 it turns out that sub-dominant terms in the initial transform only give rise to sub-dominant terms in the result. From now on the sub-dominant singularities will be neglected.

The features of the transforms that are to be tracked through the calculation will be their behaviour close to $z = 0$, the positions of the poles and the dominant behaviour there. For any transform of interest, these can be summarised thus

$$F_i(z) \sim \left\{ \begin{array}{cc} z^{m_i}, & z \sim 0 \\ A_i(1-\alpha z)^{-\beta_i}, & z \sim 1/\alpha \end{array} \right. \tag{21}$$

for an input transform like (18).

Note that all of these are also only singular at $z = 1/\alpha$. Why is this ? Firstly, differentiation of $\tilde{R}$ and the use of (7) cannot cause the transform to become non-analytic at any other point. Sub-dominant terms also remain sub-dominant. Secondly, multiplication and the use of the convolution integral (6) have much the same effect. Consider where $F_1(z)$ and $F_2(z)$ in (6) are of the form (21). How does $F_3(z)$ behave ? For $z < 1/\alpha$ the integrand is finite and so is $F_3(z)$. But as $z \to 1/\alpha$ the integrand begins to diverge at both ends of its interval and hence so can $F_3(z)$ as $z \to 1/\alpha$. For $z > 1/\alpha$, a principal value is taken in the integral where necessary (see Appendix) and $F_3(z)$ is finite. The only point where $F_3(z)$ is non-analytic is $z = 1/\alpha$. This is essentially the observation made by 't Hooft [11]. Furthermore, if $F_1(z)$ and $F_2(z)$ are as in (21), so is $F_3(z)$, but with

$$m_3 = m_1 + m_2 + 1 \tag{22}$$

$$\beta_3 = \left\{ \begin{array}{ll} -\beta_1 + m_2 + 1 & \text{if} \quad -\beta_1 + m_2 + 1 < -\beta_2 + m_1 + 1 \\ -\beta_2 + m_1 + 1 & \text{if} \quad -\beta_2 + m_1 + 1 < -\beta_1 + m_2 + 1 \end{array} \right. \tag{23}$$

One has to know the behaviours near $z = 0$ because when one of the transforms in the integrand of (6) is diverging to give the divergence in $F_3(z)$, the other transform's argument is tending towards $z = 0$. All this is neglecting the sub-dominant terms. However, such terms in either $F_1(z)$ or $F_2(z)$ do not produce dominant terms in $F_3(z)$ as a result of this convolution; they are safely neglected. Since differentiation and multiplication are the only operations involved in finding (19), its transform and any intermediate ones involved in finding it will indeed thus behave as (21).

In addition, one can use (7), (22) and (23) to find the A_i, β_i and m_i of these transforms. Most of the terms (19) turn out not to contribute to the dominant singularity in the transform of $a(R)$. At each order in λ there is

one term of the form

$$\lambda^n \tilde{R}^{n-1} \frac{d^{n-1}\tilde{R}}{dx^{n-1}} \tag{24}$$

and it is only these that contribute to this singularity. Working through the (tedious) details and summing the λ-expansion one finds that, for $\tilde{R}(a)$ given by (18), the dominant part of the transform for $a(R)$ is

$$F(z) \sim \frac{-\lambda A e^{-\lambda r_1/\alpha}}{(1-\alpha z)^\beta}, \qquad z \sim 1/\alpha, \tag{25}$$

where r_1 is the one-loop coefficient of $R(a)$. Finally, λ is set equal to one.

The main effect that inversion has had on the transform is thus to change the overall constant (the A_i in (21)). The β_i doesn't change; this need not be the case for other operations. Most importantly, the position of the singularity hasn't changed. In retrospect this is an obvious consequence of the non-obvious fact that inversion can be reduced to multiplication and differentiation.

Additional singularities at other positions do not change this basic picture. If (18) is generalised to

$$F(z) \sim \sum_m \frac{A_m}{(1-\alpha_m z)^{\beta_m}}, \qquad z \sim 1/\alpha_m, \tag{26}$$

(25) becomes

$$F(z) \sim \sum_m \frac{-A_m e^{-r_1/\alpha_m}}{(1-\alpha_m z)^{\beta_m}}, \qquad z \sim 1/\alpha_m. \tag{27}$$

The additional complications (26) introduces are all sub-dominant in (27).

The idea of a split (9) in one of the functions leading to a λ-expansion also renders operations 3 and 4 tractable. Indeed in these cases the λ-expansion appears much more familiar.

$$R_1(x + \lambda\tilde{R}_2(x)) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \tilde{R}_2(x)^n \frac{d^n R_1}{dx^n} \tag{28}$$

$$\frac{R_1(x)}{x + \lambda\tilde{R}_2(x)} = \frac{R_1(x)}{x} \sum_{n=0}^{\infty} \Big(-\frac{\lambda\tilde{R}_2(x)}{x}\Big)^n. \tag{29}$$

However since these operations involve two functions, the details and the results are contingent on the specifics of two input transforms and a discussion

is again deferred [13]. A full set of results covers the situations one is liable to be confronted with in practice.

A general feature is however clear. Because all these operations can be reduced to multiplication and differentiation, if the initial transforms have the renormalon structure predicted by QCD [1][2], so do the transforms produced by the operations. The universality of that structure is further confirmed: all QCD observables look the same in the Borel plane.

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Appendix

Although it is a standard theorem [15] that the Cauchy product of two Borel summable series is also Borel summable, the result for products required here is that the Borel transform of the product is given by the convolution (6), even when singularities are present. Since we know of no proof of this in the literature, one is outlined here. Consider two transform integrals

$$f(a) = \int_0^\infty F(z) e^{-z/a} dz, \qquad g(a) = P\int_0^\infty G(z) e^{-z/a} dz, \tag{30}$$

where $F(z)$ is analytic, but $G(z)$ has a singularity at $z = \eta$. The P indicates a principal value. The convolution integral will thus be

$$H(z) = P\int_0^z G(w) F(z-w) dw. \tag{31}$$

Now

$$\begin{aligned} h(a) &= g(a) f(a) && (32)\\ &= \lim_{\epsilon\to 0}\Big(\int_0^{\eta-\epsilon} G(z) e^{-z/a} dz + \int_{\eta+\epsilon}^\infty G(z) e^{-z/a} dz\Big) f(a) && (33)\\ &= \overline{h}(a) + \lim_{\epsilon\to 0}\int_0^{\eta-\epsilon} dz G(z) e^{-z/a}\int_{\eta-z-\epsilon}^{\eta-z+\epsilon} dw F(w) e^{-z/a}. && (34) \end{aligned}$$

where

$$\overline{h}(a) \equiv P \int_0^\infty H(z) e^{-z/a} dz. \tag{35}$$

It is the final term in (34) that is at issue. The details of whether or not it tends to zero depend on the specific $G(z)$. It suffices to consider whether

$$\epsilon \int_0^{\eta-\epsilon} G(z) dz \tag{36}$$

vanishes as $\epsilon \longrightarrow 0$. If it does, then the natural generalisation of the convolution theorem, namely that $h(a) = \overline{h}(a)$, is true.

If, as if in QCD,

$$G(z) \sim (\eta - z)^{-\beta}, \qquad z \sim \eta, \tag{37}$$

then (36) vanishes only if $\beta < 2$. However the naive principal value definition in (30) only holds for $\beta \leq 1$ anyway. For $\beta > 1$, one can define the transform via

$$g(a) = a^{-n} P \int_0^\infty \overline{G}(z) e^{-z/a} dz \tag{38}$$

for some n such that $\overline{G}(z)$ has a singularity with $\beta' = \beta - n < 1$. In this paper, the transforms are implicitly defined like this. For clarity, the a^{-n} factors are suppressed and $\beta > 1$ ia allowed, but with care it can be arranged such that $\beta < 1$ in all transforms. The convolution theorem thus holds for all products considered here.

The generalization to both F and G having multiple renormalon singularities is straightforward.

Principal values have been used to define the Borel integrals because the generalization of the theorem, (31) and (35), is then particularly natural. However it is more common in QCD to define Borel integrals using contours that detour around singularities. These versions have the disadvantage here that the contours required for the equivalents of (31) and (35) are then not obvious.

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